Lecture 44

Remaining office hours:

12/10 (today): ~~4:30~~ 5:00–6:00pm (Hurley 295)

12/13: 2:00–3:00pm
12/14: 3:00–4:00pm
12/16: 8:00–9:00pm
12/17: 4:00–5:00pm

} (Hurley 258)

- Let $\vec{F} = \langle P, Q, R \rangle$
  - $\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z}$
  - $\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$
- Be able to parametrize surfaces
  - Often, a useful tactic is to parametrize a surface is to describe it in a coordinate system (cartesian, cylindrical, spherical) where you can solve for one variable in terms of the others (or one of them become constant)

- In the case that the surface is the graph of a function, say $g(x,y)$, that is, the surface is given by the equation $z=g(x,y)$ where (x,y) is in a region D, the surface can quickly be parametrize by $\vec{r}(x,y)=\langle x,y,g(x,y)\rangle$, $(x,y)\in D$.

• Given a parametrization $\vec{r}(u,v)$, $(u,v)\in D$, of a surface S:

1) a continuous (not necessarily unit) normal vector field on the surface is given by $\vec{r}_u \times \vec{r}_v$

2) the tangent plane to the surface at a point $P_0=\vec{r}(u_0,v_0)$ has normal vector $\vec{r}_u(u_0,v_0)\times\vec{r}_v(u_0,v_0)$

3) To integrate a function $f=f(x,y,z)$ over the surface parametrized by $\vec{r}(u,v)$ we use

$$\iint_S f\,dS=\iint_D f(\vec{r}(u,v))\,|\vec{r}_u\times\vec{r}_v|\,dA$$

4) An orientation on a surface is a choice of a unit normal vector field on it

Given a parametrization as above, the two unit normal vector fields on S are given by

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \quad \text{and} \quad -\vec{n} = \frac{\vec{r}_v \times \vec{r}_u}{|\vec{r}_u \times \vec{r}_v|}$$

5) If the surface S is parametrized as above and has orientation given by $\vec{n}$ (as in (4)), then the flux of the vector field $\vec{F}$ across S is given by

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, dA$$

6) The surface area of a surface S is given by

$$A(S) = \iint_S dS$$

7) The mass of a surface S with density function ρ is $\quad m = \iint_S \rho\, dS$

8) In the special case that S is given by $z = g(x,y)$, $(x,y) \in D$, the surface integral of $f = f(x,y,z)$ over S is

$$\iint_S f\,dS = \iint_D f(x,y,g(x,y))\sqrt{(g_x)^2+(g_y)^2+1}\,dA$$

and if S has the upward orientation, the flux of $\vec{F} = \langle P, Q, R\rangle$ across S is

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D (-P g_x - Q g_y + R)\,dA$$

Let S be a surface with boundary C. If S has orientation $\vec{n}$, to get the induced orientation on C, we walk around C in the direction such that if our head points in the direction of $\vec{n}$, the surface is on the left. Likewise, we can find an orientation on S given one on C by reversing the process above.

Stokes' Theorem: Let S be an oriented surface with boundary C and give C the orientation induced by S. (C needs to be a closed curve.) Then

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\text{curl}\,\vec{F})\cdot d\vec{S}$$

Divergence Theorem: Let E be a solid region in $\mathbb{R}^3$ with boundary S and give S the outward orientation. Then

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E (\text{div}\,\vec{F})\,dV.$$